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# THE DISTRIBUTION FUNCTION OF THE CONVOLUTION SQUARE OF A CONVEX SYMMETRIC BODY IN $\mathbb{R}^n$

BY

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ABSTRACT

**By analyzing the distribution function of the convolution square of a convex and symmetric body we obtain some affine invariants related to the body. These invariants have a geometric interpretation.**

## Introduction and notations

The starting point of our investigation is a paper of K. Kiener [K]. Before we explain his results we have to introduce some notation. Let $C$ be a convex body in $\mathbb{R}^n$ (i.e. $C$ is a compact convex subset of $\mathbb{R}^n$ with non-empty interior). By $I_C$ we denote the indicator function of $C$; the convolution square of $C$ is defined by $F = I_C * I_C$ (we will also investigate the function $G = I_C * I_{-C}$ which in the case of a symmetric body coincides with $F$). The distribution function of $F$ is given by

$$V_F(\delta) = \operatorname{Vol}_n([F > \delta]) = \operatorname{Vol}_n(\{x \in \mathbb{R}^n : F(x) > \delta\}$$

where $\operatorname{Vol}_n$ denotes the $n$-dimensional Lebesgue measure. By a volume preserving linear transformation we mean a linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det T = 1$. In [K] Kiener proved the following theorem:

Let $C$ be a convex body in $\mathbb{R}^n$. Choose $\alpha > 0$ such that $\operatorname{Vol}_n(C) = \operatorname{Vol}_n(\alpha B_n^2)$ (where $B_n^2$ denotes the euclidean ball of radius 1). If the distribution function of the convolution square coincides with that of $\alpha B_n^2$ then $C$ is an ellipsoid, i.e. $C$ is an image of $\alpha B_n^2$ under a volume preserving linear transformation.

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A crucial point in proving this theorem was the following formula

$$\lim_{\delta \to \text{Vol}_n(C)} \frac{V_F(\delta)}{(\text{Vol}_n(C) - \delta)^n} = \text{Vol}_n(P^*)$$

where $V_F$ denotes as above the distribution function of the convolution square of $C$ and $P^*$ denotes the polar of the projection body of $C$. We deduce this formula from an exponential bound for the convolution square. We also analyze the behavior of $V_F(\delta)$ for symmetric convex bodies as $\delta$ tends to zero. It turns out that there is an analogy between certain bodies associated with the convolution square of a convex symmetric body and the so-called floating bodies. The corresponding results for the floating bodies were obtained by V.D. Milman and M. Gromov [G.M], C. Schütt and E. Werner [S.W] and C. Schütt [S].

I would like to thank K. Kiener for his encouragement and the referee for the many suggestions, corrections and remarks, which clarified the subject.

## The convolution square

Let $C$ be a convex symmetric body in $\mathbb{R}^2$ and let $pr_1 C = [-c, c]$ denote the projection of $C$ to the first coordinate. Define

$$f : [-c, c] \to \mathbb{R} \quad \text{by} \quad f(x) = \sup\{y : (x, y) \in C\}.$$

Then $f$ is concave and

$$C = \Big\{(x, y) \in \mathbb{R}^2 : x \in [-c, c], -f(-x) \le y \le f(x)\Big\}.$$

For $\lambda \ge 0$ set

$$x_\lambda := \max\Big\{x \ge 0 : \big(x, f(x)\big) \in \partial C \cap \big(\partial C + \lambda(0,1)\big)\Big\}.$$

LEMMA 1: *Let $\lambda$, $\lambda_0 \ge 0$, $\lambda + \lambda_0 \le 2f(0)$. Then*

$$0 \le x_{\lambda_0} - x_{\lambda_0 + \lambda} \le \lambda \frac{x_{\lambda_0}}{2f(0) - \lambda_0}.$$

*Proof:* For $t \ge 0$, $x_t$ satisfies the equation

$$f(x_t) = -f(-x_t) + t.$$

Since the function $F(x) := f(x) + f(-x)$ is concave and symmetric, the right hand side follows immediately. The left hand side of the inequality follows from the fact that $F$ is decreasing on $[0, c]$. ∎

LEMMA 2: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$. For $x_0 \in S^{n-1}$ let $\lambda, \lambda_0 \geq 0$ be such that $\lambda + \lambda_0 \leq 2/\|x_0\|_C$. Let $\sigma(C, x_0)$ denote the $(n-1)$-dimensional volume of the projection of $C$ to the hyperplane orthogonal to $x_0$. Then we have:*

$$Vol_n\, C \cap (C + \lambda_0 x_0) - Vol_n\, C \cap (C + (\lambda + \lambda_0)x_0)$$
$$\geq \lambda\sigma\big(C \cap (C + \lambda_0 x_0), x_0\big) - \lambda^2 c(\lambda_0, x_0)$$

*where* $c(\lambda_0, x_0) = \frac{n-1}{2M-\lambda_0}\sigma\big(C \cap (C + \lambda_0 x_0), x_0\big)$ *and* $M = \frac{1}{\|x_0\|_C}$.

*Proof:* We obviously have

$$\mathrm{Vol}_n\, C\cap(C+\lambda_0 x_0) - \mathrm{Vol}_n\, C\cap\big(C+(\lambda+\lambda_0)x_0\big) \geq \lambda\sigma\Big(C\cap\big(C+(\lambda+\lambda_0)x_0\big), x_0\Big).$$

The Quermaß on the right hand side can be computed by the formula

$$\sigma = \frac{1}{n-1}\int_{S^{n-2}} x_{\lambda_0+\lambda}(\xi)^{n-1}\, d\xi$$

where $x_t(\xi)$ has the previously defined meaning with respect to the 2-dimensional slice

$$C \cap \text{ span } \{x_0, \xi\}.$$

According to Lemma 1 and Bernoulli's inequality we get

$$\sigma \geq \frac{1}{n-1}\int_{S^{n-2}} \left(x_{\lambda_0}(\xi) - \lambda\frac{x_{\lambda_0}(\xi)}{2M-\lambda_0}\right)^{n-1} d\xi$$
$$\geq \sigma\big(C \cap (C + \lambda_0 x_0), x_0\big)\left(1 - \frac{n-1}{2M-\lambda_0}\lambda\right).$$

The purpose of the following observations is to improve the right hand estimate of the preceding Lemma. By $C_{\lambda, x_0}$ we denote the convex body $C \cap (C + \lambda x_0)$. ∎

LEMMA 3: *Let $C$ be a convex body in $\mathbb{R}^n$. Then the one parameter family $\lambda \mapsto C_{\lambda.x_0}$ is concave for all $x_0 \in S^{n-1}$.*

*Proof:* It is easy to show that for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$

$$C_{\alpha t+\beta s, x_0} \supseteq \alpha C_{t,x_0} + \beta C_{s,x_0}. \quad ∎$$

LEMMA 4: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$ with volume 1. For $x_0 \in S^{n-1}$ define*

$$\psi_{x_0}(t) = \mathit{Vol}_n(C_{t,x_0}).$$

*Then for all $t \in \mathbb{R}$*

$$\psi_{x_0}(t) \leq \exp\big(-|t|\sigma(C, x_0)\big).$$

*Proof:* W.l.o.g. we may assume that $t \geq 0$. By Lemma 3 and the theorem of Brunn the function

$$t \mapsto \psi_{x_0}^{1/n}(t)$$

is concave. Hence for all $\lambda \in [0,1]$

$$\begin{aligned}
\psi_{x_0}(\lambda t) &\geq \big((1-\lambda)\psi_{x_0}(0)^{1/n} + \lambda\psi_{x_0}(t)^{1/n}\big)^n \\
&= \big(1 - \lambda + \lambda\psi_{x_0}(t)^{1/n}\big)^n \\
&= \Big(1 + \lambda\big(\psi_{x_0}(t)^{1/n} - 1\big)\Big)^n \\
&= \left(1 + \lambda\Big(\exp\big(\frac{1}{n}\log\psi_{x_0}(t)\big) - 1\Big)\right)^n \\
&\geq \Big(1 + \lambda\big(1 + \frac{1}{n}\log\psi_{x_0}(t) - 1\big)\Big)^n \\
&= \big(1 + \frac{\lambda}{n}\log\psi_{x_0}(t)\big)^n.
\end{aligned}$$

Since the inequality is trivial for $\psi_{x_0}(t) = 0$, we suppose that $\psi_{x_0}(t) > 0$ and choose $\lambda \geq 0$ so that

$$\frac{\lambda}{n}\log\psi_{x_0}(t) \geq -1.$$

Under these assumptions we get using Bernoulli's inequality

$$\psi_{x_0}(\lambda t) \geq 1 + \lambda\log\psi_{x_0}(t).$$

On the other hand we get from Lemma 2

$$\psi_{x_0}(\lambda t) \leq 1 - \lambda t\sigma(C, x_0)\big(1 - \lambda tc(x_0)\big).$$

Hence

$$\log\psi_{x_0}(t) \leq -t\sigma(C, x_0)\big(1 - \lambda tc(x_0)\big).$$

Since this inequality holds for all $\lambda$ sufficiently close to zero, we can take $\lambda$ to be 0. Thus

$$\psi_{x_0}(t) \leq \exp\big(-t\sigma(C, x_0)\big).$$

*Remark:* The same proof yields the inequality

$$\psi_{x_0}(t+t_0) \leq \psi_{x_0}(t_0) \exp\left(-t\frac{\sigma(C_{t_0,x_0}, x_0)}{\psi_{x_0}(t_0)}\right)$$

which is valid for an arbitrary convex symmetric body $C$. Putting all together we get the following

COROLLARY 1: *Let $C$ be a convex symmetric body in $\mathbf{R}^n$. Then for all $x \in S^{n-1}$ and all $t \geq 0$*

$$1 - t\frac{\sigma(C,x)}{Vol_n(C)} \leq \frac{\psi_x(t)}{Vol_n(C)} \leq \exp\Big(-t\frac{\sigma(C,x)}{Vol_n C}\Big)$$

*equivalently, for all $x \in \mathbb{R}^n$:*

$$1 - \frac{\|x\|_{P^*}}{Vol_n(C)} \leq \frac{G(x)}{Vol_n(C)} \leq \exp\Big(-\frac{\|x\|_{P^*}}{Vol_n(C)}\Big)$$

*where $G$ denotes the convolution $I_C * I_{-C}$ and $P^*$ the polar of the projection body of $C$.*

*Remark:* The corollary remains true if we only assume $C$ to be a convex body. This is because Lemma 2 is true in this context up to another factor $c(x)$, whose explicit value is not relevant in the proof of Lemma 4 – it suffices that it be positive. ∎

We are now going to apply Corollary 1 to the convolution square of a convex (symmetric) body.

THEOREM 1: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$ with volume 1 and set*

$$C(\delta) = \big\{x \in \mathbb{R}^n : \ Vol\ C \cap (C + x) \geq \delta\big\} \qquad (0 \leq \delta \leq 1).$$

*Then $C(\delta)$ is a convex symmetric body and*

$$(1-\delta)P^* \subseteq C(\delta) \subseteq \log\Big(\frac{1}{\delta}\Big)P^*.$$

*Proof:* The first assertion, which was already proved in [M], follows from the Brunn–Minkowski inequality and the fact that

$$(1-\lambda)\big(C\cap(C+x)\big)+\lambda\big(C\cap(C+y)\big)\subseteq C\cap\big(C+(1-\lambda)x+\lambda y\big).$$

Now let $x$ be in $S^{n-1}$. Then for $p(x):=\frac{1}{\|x\|_{C(\delta)}}$ we have

$$\mathrm{Vol}_n\, C\cap\big(C+p(x)x\big)=\delta.$$

By Corollary 1 we get for $t=p(x)$

$$1-\sigma(C,x)p(x)\le\delta\le\exp\big(-p(x)\sigma(C,x)\big)$$

$$\Longleftrightarrow \|x\|_{C(\delta)}(1-\delta)\le\sigma(C,x)\le\log\frac{1}{\delta}\|x\|_{C(\delta)}.$$

This is the desired inequality for the associated norms. ∎

Corollary 2: *Let $x$ be in $S^{n-1}$, $t\ge 0$ and $F(tx)=\psi_x(t)$ the convolution square of a convex symmetric body $C$ with volume 1. Let $P^*$ be the polar of the projection body of $C$ and $V_F(\delta):=\ \mathit{Vol}_n\big([F>\delta]\big)$, the distribution function of $F$. Then*

$$(1-\delta)^n\ \mathit{Vol}_n\,P^*\le V_F(\delta)\le(\log\frac{1}{\delta})^n\ \mathit{Vol}_n\,P^*.$$

Taking the limit as $\delta\to 1$ we get the above mentioned theorem of Kiener:

$$\lim_{\delta\to 1}\frac{V_F(\delta)}{(1-\delta)^n}=\ \mathrm{Vol}_n\,P^*.$$

In fact we get something more:

Corollary 3: $\lim\limits_{\delta\to 1}(1-\delta)^{-1}C(\delta)=P^*$ *in the Hausdorff-metric.*

## The affine surface area

We next recall the notion of the floating body, more exactly the convex floating body of a convex body $C$. Both concepts coincide in the case of a convex symmetric body $C$ as was proved independently by K. Ball (unpublished) and by M. Meyer and S. Reisner [M.S]. We repeat the definition of [S.W].

*Definition:* The convex floating body $C_\delta$ of a convex body $C$ in $\mathbb{R}^n$ is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume $\delta$ from the set $C$. If $A$ denotes the set of all pairs $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\operatorname{Vol}_n \left\{y \in C : \langle y, x\rangle \geq t\right\} = \delta$$

then

$$C_\delta = \bigcap_{(x,t)\in A} \left\{y \in \mathbb{R}^n : \langle y, x\rangle \leq t\right\}.$$

Gromov and Milman [G.M] (see also [M.P]) proved that for any convex symmetric body $C$ in $\mathbb{R}^n$ the floating body $C_\delta$ is isomorphic to the Legendre Ellipsoid $L(C)$ and the constant of isomorphism does not depend on $n$ (i.e. for all $0 < \delta < \operatorname{Vol}_n(C)/2$ there exists a constant $c(\delta)$ such that $c(\delta)^{-1}L(C) \subseteq C_\delta \subseteq c(\delta)L(C)$). The proof depends on a concentration property which plays the same role as Lemma 4 in the proof of Theorem 1. Actually Lemma 4 was proved in the spirit of this concentration property.

Our next aim is to set up an analogy (cf. [B.L] for both concepts) between the convex floating bodies $C_\delta$ and the bodies $C(\delta)$ by showing that the affine surface area of a convex and symmetric body $C$ can also be defined via the distribution function of the convolution square of $C$. However, we need some definitions, lemmata and classical results.

*Definition:* The affine surface area of a convex body $C$ in $\mathbb{R}^n$ is defined by

$$S_{aff}(C) = \lim_{\delta\to 0} c_n \frac{\operatorname{Vol}_n C - \operatorname{Vol}_n C_\delta}{\delta^{2/(n+1)}}$$

where

$$c_n = 2\left(\frac{w_{n-1}}{n+1}\right)^{\frac{2}{n+1}},$$

where the symbol $w_n$ denotes the volume of the unit ball $B_n^2$ of $l_n^2$. ∎

For $x \in \partial C$, the outer normal $N(x)$, $\|N(x)\|_2 = 1$, exists almost everywhere. If $\Delta(x,\delta)$ denotes the width of the slice

$$\left\{y \in C : \big\langle y, N(x)\big\rangle \geq \big\langle x, N(x)\big\rangle - \Delta(x,\delta)\right\}$$

of volume $\delta$, then, as was shown in [S.W], the affine surface area can be computed as an integral

$$S_{aff}(C) = \int_{\partial C} \lim_{\delta\to 0} c_n \frac{\Delta(x,\delta)}{\delta^{2/(n+1)}}\, d\lambda(x)$$

where $\lambda$ is the Lebesgue measure on $\partial C$.

By the formula of Schütt and Werner we obtain for the affine surface area of a Euclidean ball of radius $r$.

$$S_{aff}(rB_n^2) = nw_n r^{\frac{n(n-1)}{n+1}}.$$

For the various definitions of the affine surface area we refer to [L], [Lu]. Also, in a very recent preprint C. Schütt proved that all of these definitions are equivalent.

For the proofs of the next three lemmata we refer to [S.W].

LEMMA 5: *Let $C_1$ and $C_2$ be convex bodies in $\mathbb{R}^n$ such that 0 is an interior point of $C_2$ and $C_2 \subseteq C_1$. Then*

$$Vol_n C_1 - Vol_n C_2 = \frac{1}{n} \int_{\partial C_1} \langle x, N(x) \rangle \left( 1 - \left( \frac{\|x'\|_2}{\|x\|_2} \right)^n \right) d\lambda(x)$$

*where $x \in \partial C_1$ and $x' \in \partial C_2$ is such that $x'$ lies on the line $[0, x]$.*

*Remark:* It is easy to see that whenever

$$\lim_{\delta \to 0} \delta^{-\frac{2}{n+1}} \langle x, N(x) \rangle \|x - x'\|_2 \|x\|_2^{-1} = \lim_{\delta \to 0} \delta^{-\frac{2}{n+1}} \langle x - x', N(x) \rangle$$

exists, then

$$\lim_{\delta \to 0} \frac{1}{n} \delta^{-\frac{2}{n+1}} \langle x, N(x) \rangle \left( 1 - \left( \frac{\|x'\|_2}{\|x\|_2} \right)^n \right)$$

also exists and both limits coincide (we assume that $x'$ converges to $x$ as $\delta$ converges to zero). ∎

LEMMA 6: *Let $C$ be a convex body in $\mathbb{R}^n$. For every $x \in \partial C$ let $r(x)$ be the radius of the largest Euclidean ball that is contained in $C$ and that contains $x$. Then for all $\alpha$ with $0 < \alpha < 1$*

$$\int_{\partial C} r(x)^{-\alpha} \, d\lambda(x) < \infty.$$

LEMMA 7: *Let $B_n^2(r, h)$ be a cap of a Euclidean ball with radius $r$ and height $h$ in $\mathbb{R}^n$. Then there is a continuous function $g$ with $\lim_{t \to 0} g(t) = \sqrt{2}$ so that for $0 < h < r$*

$$Vol_n B_n^2(r, h) = g\left(\frac{h}{r}\right)^{n+1} \frac{w_{n-1}}{n+1} h^{\frac{n+1}{2}} r^{\frac{n-1}{2}}.$$

LEMMA 8: *Let $C$ be a convex body in $\mathbb{R}^n$, $x_0 \in \partial C$, $0 \in \overset{\circ}{C}$.*

(i) *If $T$ is a linear isomorphism and $N(x_0)$ is the normal at $x_0$, then $T^{*-1}N(x_0)$ is a normal at $Tx_0$.*

(ii) *There exists a linear volume preserving transformation $T$ such that $Tx_0$ and $T^{*-1}N(x_0)$ are collinear and $\|T^{*-1}N(x_0)\|_2 = 1$. Moreover, this transformation does not affect the Gauss–Kronecker curvature (if it exists) of $C$ at $x_0$, i.e. $k_{TC}(Tx_0) = k_C(x_0)$.*

(iii) *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear volume preserving transformation, then $(TC)(\delta) = T(C(\delta))$.*

*Proof:* The proofs are straightforward. ∎

LEMMA 9: *Let $E$ be the ellipsoid*

$$\left\{x \in \mathbb{R}^n : \sum_{i=1}^{n}\left(\frac{x_i}{r_i}\right)^2 \le 1\right\}.$$

*Then*

$$\int_E \sum_{i=1}^{m} x_i^2\, dx = \frac{\mathrm{Vol}_n E}{n+2}\Big(\sum_{i=1}^{m} r_i^2\Big) \qquad (1 \le m \le n).$$

*Proof:* Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $e_i \mapsto r_i e_i$. Then $TB_n^2 = E$ and

$$\int_E \langle x, e_i\rangle^2\, dx = \int_{B_n^2} r_i^2 |\det T| \langle x, e_i\rangle^2\, dx = \frac{|\det T|}{n} r_i^2 \int_{B_n^2} \|x\|_2^2\, dx$$

$$= r_i^2 \frac{\mathrm{Vol}_n E}{\mathrm{Vol}_n B_n} \frac{1}{n+2}\ \mathrm{Vol}_n B_n^2.$$

Hence

$$\int_E \sum_{i=1}^{m} x_i^2\, dx = \frac{\mathrm{Vol}_n E}{n+2}\Big(\sum_{i=1}^{m} r_i^2\Big). \qquad ∎$$

Now recall the definitions of $C(\delta)$ and $C_\delta$. Let $x$ be in $\frac{1}{2}\partial C(2\delta)$ and let $K(x) = C \cap (C + 2x)$. Then $K(x)$ is symmetric with respect to $x$. Hence every hyperplane $H$ passing through $x$ cuts off a subset of volume $\frac{1}{2}\ \mathrm{Vol}_n K(x)\,(= \delta)$ from the set $K(x)$. Since $K(x) \subseteq C$ we get

$$\mathrm{Vol}_n (C \cap H^+) \ge \delta.$$

Therefore (cf. [B.L] )

$$\frac{1}{2}C(2\delta) \subseteq C_\delta.$$

In case $n = 2$, it turns out that both bodies coincide: Let $x_1$ be in $\partial C \cap (\partial C + 2x)$ but distinct from $x$. Then there exists a line $g$ passing through both $x$ and $x_1$ and cutting off a segment of area $\delta$ from $C$. Since $x$ is the barycenter of $C \cap g$ (here we use the symmetry of $C$), $x$ must be in $\partial C_\delta$.

In the general case the next Lemma is important (compare [S.W.] Lemma 6).

LEMMA 10: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$, $x \in \partial C$. Let $x_\delta$ be the unique element with $2x_\delta \in \partial C(2\delta)$ and $x_\delta \in [0, x]$. Then we have the following estimates:*

(i) $$\mathit{Vol}_n\, C \cap (C + 2x_\delta) \geq (1 - \|x_\delta\|_C)^n\ \mathit{Vol}_n\, C.$$

(ii) $$\mathit{Vol}_n\, C \cap (-C + 2x_\delta) \geq \left(\frac{\|x - x_\delta\|_2}{\|x\|_2}\right)^n \alpha^{-n} w_n.$$

(iii) *If* $\|x - x_\delta\|_2 \leq \frac{1}{\alpha^2} r(x)$ *then*

$$\mathit{Vol}_n\, C \cap (-C + 2x_\delta) \geq 2\ \mathit{Vol}_n\, B_n\left(r(x), \frac{1}{2}\frac{\|x - x_\delta\|_2}{\alpha^2}\right).$$

*(In (ii) and (iii) we assume that $\alpha^{-1} B_n^2 \subseteq C \subseteq \alpha B_n^2$. Both statements still hold in the non-symmetric case.)*

*Proof:*

(i) It is easy to check

$$C \cap (C + 2x_\delta) - x_\delta \supseteq (1 - \|x_\delta\|_C) C.$$

(ii) Define

$$K = co(x, \frac{1}{\alpha} B_n) \subseteq C.$$

Then

$$K \cap (-K + 2x_\delta) \subseteq C \cap (C + 2x_\delta) \qquad \text{and}$$

$$\mathrm{Vol}_n\, (K \cap (-K + 2x_\delta)) \geq \left(\frac{\|x - x_\delta\|_2}{\|x\|_2}\right)^n \alpha^{-n} w_n.$$

(iii) From the figure below it follows that

$$l^2 = \|x - x_\delta\|_2^2 + r(x)^2 - 2r(x)\|x - x_\delta\|_2 \cos\theta,$$

$$\cos\theta = \langle x, N(x)\rangle \|x\|_2^{-1} \geq \alpha^{-2}.$$

Assuming $l \leq r(x)$, which is true when $r(x) > 0$ and $\|x - x_\delta\|_2$ is small enough, we get

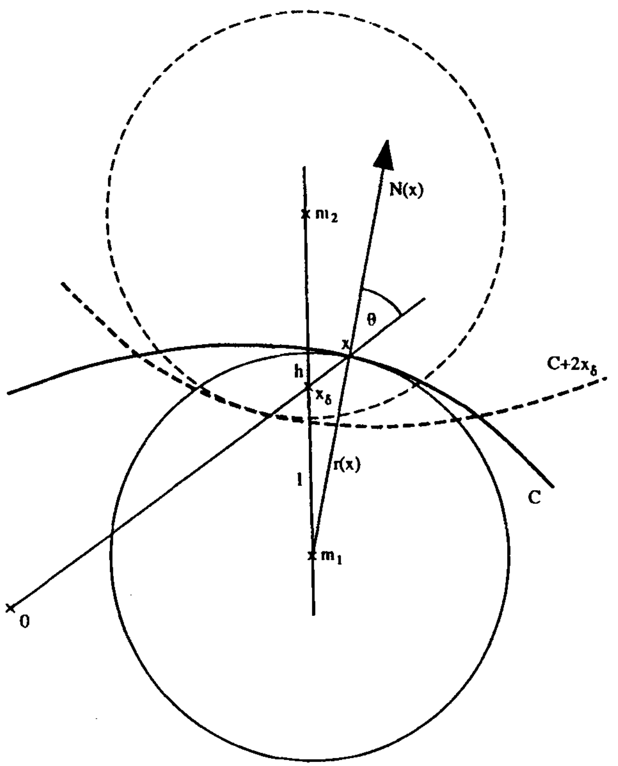
$$h\big(r(x) + l\big) = \big(r(x) - l\big)\big(r(x) + l\big) = r^2(x) - l^2$$

$$= -\|x - x_\delta\|_2^2 + 2\|x - x_\delta\|_2 r(x)\cos\theta;$$

we conclude that

$$h \geq \frac{\|x - x_\delta\|_2}{2r(x)}\big(2r(x)\alpha^{-2} - \|x - x_\delta\|_2\big) \geq \frac{1}{2}\frac{\|x - x_\delta\|}{\alpha^2}.$$

Hence $C \cap (-C + 2x_\delta)$ contains the cap $B_n^2\left(r(x), \frac{1}{2}\frac{\|x - x_\delta\|_2}{\alpha^2}\right)$.



COROLLARY 4: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$. Then for all $x \in \partial C$ with $r(x) > 0$*

$$\delta^{-\frac{2}{n+1}} \langle x - x_\delta, N(x) \rangle \leq c r(x)^{-\frac{n-1}{n+1}}$$

*where c is a constant depending only on $C$ and $n$.*

*Proof:* $\delta^{-\frac{2}{n+1}} \langle x - x_\delta, N(x) \rangle \leq \delta^{-\frac{2}{n+1}} \|x - x_\delta\|_2$.
If $\|x - x_\delta\|_2 \geq \frac{1}{\alpha^2} r(x)$ we get by Lemma 10 (ii):

$$\delta \geq \left( \frac{\|x - x_\delta\|_2}{\|x\|_2} \right)^n \alpha^{-n} w_n \geq c_1 \|x - x_\delta\|^n.$$

Hence

$$\|x - x_\delta\|_2 \delta^{-\frac{2}{n+1}} \leq c_2 \|x - x_\delta\|_2^{1 - \frac{2n}{n+1}} \leq c_3 r(x)^{-\frac{n-1}{n+1}}.$$

If $\|x - x_\delta\|_2 \leq \frac{1}{\alpha^2} r(x)$ then it follows from Lemma 7 and Lemma 10 (iii) that

$$\begin{aligned} \delta &\geq \operatorname{Vol}_n \left( B\left( r(x), \frac{1}{2} \frac{\|x - x_\delta\|_2}{\alpha^2} \right) \right) \\ &\geq c_4 \|x - x_\delta\|_2^{\frac{n+1}{2}} r(x)^{\frac{n-1}{2}}. \end{aligned}$$

Now we get

$$\begin{aligned} \|x - x_\delta\|_2 \delta^{-\frac{2}{n+1}} &\leq c_5 \|x - x_\delta\|_2^{-1} r(x)^{-\frac{n-1}{n+1}} \|x - x_\delta\|_2 \\ &= c_5 r(x)^{-\frac{n-1}{n+1}}. \end{aligned}$$

∎

*Definition:* Let $\varphi : U \to \mathbb{R}$ be a convex function on an open convex subset $U$ of $\mathbb{R}^n$. We say that $\varphi$ is twice differentiable (in a generalized sense) at $x_0 \in U$ if there exists a linear map $d^2\varphi(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ so that for all $x$ in a neighborhood $U(x_0)$ of $x_0$ and all subdifferentials $d\varphi(x)$

$$\|d\varphi(x) - d\varphi(x_0) - d^2\varphi(x_0)(x - x_o)\|_2 \leq \Delta(\|x - x_0\|_2)\|x - x_0\|_2$$

where $\Delta : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function with $\lim_{t \to 0} \Delta(t) = 0$ (cf. [Ban], [S]). ∎

THEOREM (Aleksandrov): *$\varphi$ is a.e. twice differentiable.*

For a proof see [Ban].

PROPOSITION: *The Hessian $H(x_0)(u,v) := \langle d^2\varphi(x_0)u, v\rangle$ is a positive symmetric form (i.e. $H(x_0)(u,u) \geq 0$) and*

$$|\varphi(x) - \varphi(x_0) - d\varphi(x_0)(x - x_0) - \frac{1}{2}H(x_0)(x - x_0, x - x_0)| \leq \tilde{\Delta}(\|x - x_0\|_2)\|x - x_0\|_2^2$$

*for all $x$ in a neighborhood $U(x_0)$.*

See [Ban] p. 321.

Using a properly chosen translation and a rotation we may assume that the boundary of a convex body $C$ is given locally by

$$x_n = \varphi(x_1, \dots, x_{n-1}),$$

that $0 \in \partial C$ and that $d\varphi(0) = 0$.

Geometrically the above proposition says that the projection of

$$\frac{1}{\sqrt{2h}}(\partial C \cap [x_n = h])$$

to the subspace orthogonal to $(0, \dots, 0, 1)$ converges radially to $\{u \in \mathbb{R}^{n-1} : H(0)(u,u) = 1\}$. The latter quadratic form is called the indicatrix of Dupin.

We now have all ingredients required for the proof of the following

PROPOSITION 1: *Let $\varphi_0$ be a local parameterization of the boundary of a convex symmetric body $C(\subseteq \mathbb{R}^n)$. Suppose that*

(1) $$\varphi_0(0) = h,$$

(2) $$d\varphi_0(0) = 0,$$

(3) $$H(0)(x,x) = -k(x_1^2 + \ldots + x_m^2), \qquad k > 0, m \leq n-1.$$

*Define*

$$\phi(x) = \varphi_0(x) + \varphi_0(-x) \qquad \text{and} \qquad \delta = \frac{1}{2}\int_{[\phi \geq 0]} \phi(x)\,dx$$

*Then*

$$\frac{h}{\delta^{2/(n+1)}} \leq \frac{2}{c_n}^{\frac{n-m-1}{n+1}} k^{\frac{m}{n+1}} \Delta(h)^{\frac{n-m-1}{n+1}} (1 + g(h))$$

*where* $\Delta$ *and* $g$ *are positive functions with* $\lim_{h\to 0}\Delta(h) = \lim_{h\to 0} g(h) = 0$.

*Proof:* For $x \in \mathbb{R}^{n-1}$ define

$$x' = (x_1, \ldots, x_m), \qquad x'' = (x_{m+1}, \ldots, x_{n-1}).$$

By the proposition above we get

$$h - \frac{k}{2}\|x'\|_2^2 - \tilde{\Delta}(\|x\|_2)\|x\|_2^2 \le \varphi_0(x) \le h - \frac{k}{2}\|x'\|_2^2 + \tilde{\Delta}(\|x\|_2)\|x_2\|^2.$$

Hence

$$(*) \qquad 2h - k\|x'\|_2^2 - 2\tilde{\Delta}(\|x\|_2)\|x\|_2^2 \le \phi(x) \le 2h - k\|x'\|_2^2 + 2\tilde{\Delta}(\|x\|_2)\|x\|_2^2.$$

Assuming $\tilde{\Delta}(\|x\|) \le \Delta$, it follows that $\phi(x) \ge 0$ whenever

$$(k + 2\Delta)\|x'\|_2^2 + 2\Delta\|x''\|_2^2 \le 2h.$$

The equation

$$(k + 2\Delta)\|x'\|_2^2 + 2\Delta\|x''\|_2^2 \le 2h$$

defines an ellipsoid $E$ in $\mathbf{R}^{n-1}$ with principal axes

$$r = \sqrt{\frac{2h}{k+2\Delta}}, \qquad R = \sqrt{\frac{h}{\Delta}}.$$

Applying Lemma 9 and the left hand side of $(*)$ we obtain:

$$\begin{aligned}
\delta = \frac{1}{2}\int_{\phi\ge 0}\phi &\ge \int_E h - \Big(\frac{k}{2} + \Delta\Big)\|x'\|_2^2 - \Delta\|x''\|_2^2\,dx \\
&= h \text{ Vol }(E) - \Big(\frac{k}{2} + \Delta\Big)\frac{\text{Vol } E}{n+1} m r^2 - \Delta\frac{\text{Vol } E}{n+1}(n-1-m)R^2 \\
&= h \text{ Vol }(E)\Big(1 - \frac{m}{n+1} - \frac{n-1-m}{n+1}\Big) \\
&= \frac{2}{n+1} h \text{ Vol } E.
\end{aligned}$$

Since

$$\text{Vol } E = \Big(\frac{2h}{k+2\Delta}\Big)^{\frac{m}{2}}\Big(\frac{h}{\Delta}\Big)^{\frac{n-m-1}{2}} w_{n-1}$$

we get

$$\delta \ge \frac{2^{\frac{m}{2}+1}}{n+1} w_{n-1} h^{\frac{n+1}{2}} (k+2\Delta)^{-\frac{m}{2}} \Delta^{-\frac{n-m-1}{2}}.$$

Hence

$$\frac{h}{\delta^{2/(n+1)}} \le \frac{2^{\frac{n-m-1}{n+1}}}{c_n} k^{\frac{m}{n+1}} \Delta^{\frac{n-m-1}{n+1}} \Big(1 + \frac{2\Delta}{k}\Big)^{\frac{m}{n+1}}. \qquad ∎$$

*Remark:* 1) If $m = n-1$ then the same method (just use the right hand side of $(*)$!) gives the inequality

$$\frac{h}{\delta^{2/(n+1)}} \geq \frac{1}{c_n} k^{\frac{n-1}{n+1}} \left(1 - \tilde{g}(h)\right).$$

This shows that $\frac{h}{\delta^{2/(n+1)}}$ converges to zero if $m < n-1$ and it converges to $c_n^{-1} k^{(n-1)/(n+1)}$ if $m = n-1$.

2) By considering $\varphi_0$ rather than $\phi$ and setting $\delta = \int\limits_{\varphi \geq 0} \varphi(x)\,dx$ we obtain the same estimates (up to irrelevant factors). This provides another proof of the result of C. Schütt and E. Werner. ∎

PROPOSITION 2: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$. Then*

$$\lim_{\delta \to 0} \int\limits_{\partial C} c_n \delta^{-\frac{2}{n+1}} \left\langle x - x_\delta, N(x) \right\rangle d\lambda(x) = \int\limits_{\partial C} k(x)^{\frac{1}{n+1}}\, d\lambda(x)$$

*where $k(x)$ is the generalized Gauss–Kronecker curvature.*

*Proof:* Fix $x \in \partial C$ such that $k(x)$ exists. By Lemma 8 the values at $x$ of the integrands of both sides are invariant under linear transformations $T$ satisfying $\|T^{*-1} N(x)\|_2 = 1$. Hence we can assume that

$$x \| N(x).$$

Applying another affine transformation on the tangent space at $Tx$ of $\partial TC$ we see that we can also assume that the indicatrix of Dupin at $x$ is a spherical cylinder. Now we are in a position to apply the preceding proposition and the remark following it. Corollary 4, Lemma 6 and Lebesgue's dominated convergence theorem imply the proposition. ∎

THEOREM 2: *Let $C$ be a convex symmetric body in $\mathbb{R}^n$. Let $V_F$ be the distribution function of the convolution square of $C$. Then*

$$\lim_{\delta \to 0} \frac{V_F(0) - V_F(\delta)}{\delta^{2/(n+1)}} = \frac{2^{n - \frac{2}{n+1}}}{c_n} S_{aff}(C).$$

*Proof:* $V_F(0) - V_F(2\delta) = 2^n \Big( \operatorname{Vol}_n C - \operatorname{Vol}_n \left(\frac{1}{2} C(2\delta)\right)\Big)$. Therefore we get, by Proposition 2 and [S.W],

$$\lim_{\delta \to 0} \frac{V_F(0) - V_F(2\delta)}{\delta^{2/n+1}} = \frac{2^n}{c_n} S_{aff}(C). \qquad ∎$$

COROLLARY 5: *A convex symmetric body of class $C^2$ and a polytope never have the same distribution function.*

*Proof:* The affine surface area of a polytope is zero! ∎

*Remark:* As was pointed out by the referee, the above Theorem holds, if we only assume $C$ to be a convex body; this is because $F(x) = I_C * I_C = \mathrm{Vol}_n(C \cap (x - C)$. Therefore it would be more natural to work with $\widetilde{C}(\delta) := \{x \in \mathbb{R}^n : \mathrm{Vol}\, C \cap (-C + x) \geq \delta\}$ instead of $C(\delta)$. However, Theorem 1 does not hold for $\widetilde{C}(\delta)$.

As far as we know, the following problem is open: Let $V_{F_1}$ ($V_{F_2}$ respectively) be the distribution function of a polytope $P_1$ ($P_2$ resp.)$\subseteq \mathbb{R}^2$. Assume $V_{F_1} = V_{F_2}$. Does this imply that $P_1 = P_2$ up to affine transformation? ∎

## Polytopes

In a recent work [Schü] C. Schütt proved the following

THEOREM: *Let $P$ be a convex polytope in $\mathbb{R}^n$ with nonempty interior. Then*

$$\lim_{\delta \to 0} \frac{Vol_n(P) - Vol_n(P_\delta)}{\delta (\log \frac{1}{\delta})^{n-1}} = \frac{1}{n!} \frac{1}{n^{n-1}} \phi_n(P)$$

*where $\phi_n(P)$ is defined as follows:*

If $n = 1$, then $\phi_1(P) = 2$.

If $n \geq 2$, then we choose for every extreme point $x$ of $P$ a hyperplane $H_x$ that separates $x$ from the remaining extreme points and set

$$\phi_n(P) = \sum_{x \in \mathrm{ext}\,(P)} \phi_{n-1}(P \cap H_x).$$

It turns out that an analogous statement with $P_\delta$ replaced by $P(\delta)$ holds. The method of proof follows Schütt's method, with some modifications.

LEMMA 11:

(i) $$Vol_n\Big(0 \leq x_j \leq 1, \prod_{j=1}^{n} x_j \leq t\Big) = \frac{1}{(n-1)!} \gamma_n\Big(\log \frac{1}{t}\Big), \qquad t \leq 1,$$

*where $\gamma_n(\alpha) = \int_\alpha^\infty r^{n-1} e^{-r}\, dr$.*

(ii) $$t\Big(\log \frac{1}{t}\Big)^{n-1} \leq \gamma_n\Big(\log \frac{1}{t}\Big) \leq t\Big(\log \frac{1}{t}\Big)^{n-1} + C(n)t\Big(\log \frac{1}{t}\Big)^{n-2} \quad \forall 0 < t \leq 1/2.$$

*Proof:* Define
$$f : (\mathbf{R}^+)^n \to (0,1]^n \qquad \text{by}$$
$$t_j \mapsto e^{-t_j}, \qquad j = 1, \ldots, n.$$
Then
$$\left|\det Df(t_1, \ldots, t_n)\right| = \exp(-\sum t_j)$$
and
$$\begin{aligned}
\mathrm{Vol}_n\,(0 \le x_j \le 1, \Pi x_j \le t) &= \int\limits_{\sum t_j \ge \log \frac{1}{t}} \exp(-\sum t_j) dt_1 \ldots, dt_n \\
&= 2^{-n} \int\limits_{\|x\|_1 \ge \log \frac{1}{t}} \exp(-\|x\|_1)\, dx \\
&= 2^{-n} \int\limits_{\log \frac{1}{t}}^{\infty} \int\limits_{S^{n-1}} r^{n-1} \|\xi\|_1^{-n} e^{-r}\, d\xi dr \\
&= 2^{-n} n \,\mathrm{Vol}_n\,(B_n^1) \int\limits_{\log \frac{1}{t}}^{\infty} r^{n-1} e^{-r}\, dr \\
&= \frac{1}{(n-1)!} \gamma_n(\log \frac{1}{t}).
\end{aligned}$$
(ii) Integration by parts gives the formula:
$$\gamma_n(\alpha) = e^{-\alpha}\alpha^{n-1} + (n-1)\gamma_{n-1}(\alpha).$$
Therefore
$$\begin{aligned}
\lim_{\alpha \to \infty} \frac{\gamma_n(\alpha)}{e^{-\alpha}\alpha^{n-1}} &= 1 + \lim_{\alpha \to \infty} \frac{n-1}{\alpha} \int\limits_{\alpha}^{\infty} (\frac{t}{\alpha})^{n-2} e^{-t+\alpha}\, dt \\
&= 1 + \lim_{\alpha \to \infty} \frac{n-1}{\alpha} \int\limits_{0}^{\infty} (1 + \frac{x}{\alpha})^{n-2} e^{-x}\, dx \\
&= 1. \qquad \blacksquare
\end{aligned}$$

As an example, we compute the distribution function of the convolution square of the cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$. It is easy to see that
$$F(x) := I_{Q_n} * I_{Q_n}(x) = \prod_{j=1}^{n} (1 - |x_j|).$$

Hence

$$V_F(\delta) := \operatorname{Vol}_n(F \geq \delta) = 2^n \operatorname{Vol}_n(0 \leq x_j \leq 1, \Pi(1-x_j) \geq \delta).$$

Using the transformation

$$f : (\mathbb{R}^+)^n \to [0,1)^n$$
$$t_j \mapsto 1 - e^{-t_j}$$

it is easily checked that

$$V_F(\delta) = \frac{2^n}{(n-1)!} \int_0^{\log \frac{1}{\delta}} r^{n-1} e^{-r}\, dr.$$

Thus

$$\lim_{\delta \to 0} \frac{V_F(0) - V_F(\delta)}{\delta (\log \frac{1}{\delta})^{n-1}} = \frac{2^n}{(n-1)!}$$

or equivalently

$$\lim_{\delta \to 0} \frac{\operatorname{Vol}_n(Q_n) - \operatorname{Vol}_n\left(\frac{1}{2} Q_n(2\delta)\right)}{\delta (\log \frac{1}{\delta})^{n-1}} = \frac{2}{(n-1)!}$$

which, of course, coincides with the expression in Schütts' theorem when $n = 2$ $(\phi_n(Q_n) = 2^n n!)$.

LEMMA 12: *Let $S$ be a simplex in $\mathbb{R}^n$ such that 0 is an extreme point of $S$. Define*

$$\widetilde{S}(2\delta) = \left\{x \in S : \ Vol_n\left(S \cap (-S + 2x)\right) \geq 2\delta\right\}.$$

*Then*

$$Vol_n\left(S \setminus \widetilde{S}(2\delta)\right) \geq \frac{1}{2^{n-1}(n-1)!} \delta \Big(\log \frac{2^{n-1} n!}{n^n} \frac{Vol_n(S)}{\delta}\Big)^{n-1}.$$

*Proof:* Since $\operatorname{Vol}_n(S \setminus \widetilde{S}(2\delta))$ is invariant under volume preserving linear transformations we may assume that the extreme points of $S$ are given by

$$0, \alpha e_1, \ldots, \alpha e_n \qquad \text{for some} \quad \alpha > 0,$$

where $(e_j)_{j=1}^n$ is the standard unit vector basis of $\mathbb{R}^n$. In this case the boundary of $\widetilde{S}(2\delta)$ is given by

$$2^n \prod_{j=1}^n x_j - 2\delta = 0.$$

Now $S$ contains the cube $W = [0, \frac{\alpha}{n}]^n$. Therefore we get, by Lemma 11:

$$\begin{aligned}\mathrm{Vol}_n\left(S \setminus \widetilde{S}(2\delta)\right) \geq & \ \mathrm{Vol}_n\left(W \setminus \widetilde{S}(2\delta)\right) \\ = & \ \mathrm{Vol}_n\,(0 \leq x_j \leq \frac{\alpha}{n}, \prod_{j=1}^{n} x_j \leq \frac{\delta}{2^{n-1}}) \\ = & \ \mathrm{Vol}_n\left(0 \leq x_j \leq 1, \prod_{j=1}^{n} x_j \leq (\frac{n}{\alpha})^n \frac{\delta}{2^{n-1}}\right) \cdot (\frac{\alpha}{n})^n \\ \geq & \ (\frac{\alpha}{n})^n (\frac{n}{\alpha})^n \frac{\delta}{2^{n-1}} \frac{1}{(n-1)!}\left(\log(\frac{\alpha^n 2^{n-1}}{n^n \delta})\right)^{n-1} \\ = & \ \frac{1}{2^{n-1}(n-1)!}\delta\left(\log \frac{2^{n-1} n!}{n^n} \frac{\mathrm{Vol}_n\,(S)}{\delta}\right)^{n-1}. \qquad \blacksquare\end{aligned}$$

LEMMA 13: *Let $S$ be as in Lemma 12. Then*

$$Vol_n\left(S \setminus \widetilde{S}(2\delta)\right) \leq \frac{1}{2^{n-1}(n-1)!}\delta\left(\log \frac{2^{n-1} n!}{n^n} \frac{Vol_n\,(S)}{\delta}\right)^{n-1} + c(n)\delta(\log \frac{1}{\delta})^{n-2}.$$

*Proof:* W.l.o.g. we may assume that

$$S = co(0, e_1, \dots, e_n).$$

By Lemma 11 we get for $W = [0, \frac{1}{n}]^n$

$$\mathrm{Vol}_n\left(W \setminus \widetilde{S}(2\delta)\right) \leq \frac{1}{2^{n-1}(n-1)!}\delta\left(\log \frac{2^{n-1}}{n^n \delta}\right)^{n-1} + c_1(n)\delta(\log \frac{1}{\delta})^{n-2}.$$

If $x_n \geq \frac{1}{n}$ is fixed, we get (from Lemma 11):

$$\begin{aligned}&\mathrm{Vol}_{\,n-1}\left(0 \leq x_j \leq 1, \sum_{j=1}^{n-1} x_j \leq 1 - x_n, \prod_{j=1}^{n-1} x_j \leq \frac{\delta}{x_n 2^{n-1}}\right) \\ &\leq \ \mathrm{Vol}_{\,n-1}\left(0 \leq x_j \leq 1 - x_n, j = 1, \dots, n-1, \prod_{j=1}^{n-1} x_j \leq \frac{\delta}{x_n 2^{n-1}}\right) \\ &= (1 - x_n)^{n-1}\ \mathrm{Vol}_{\,n-1}\left(0 \leq x_j \leq 1, \prod_{j=1}^{n-1} x_j \leq \frac{\delta}{x_n(1-x_n)^{n-1}2^{n-1}}\right) \\ &\leq \begin{cases} c_2(n)\frac{\delta}{x_n 2^{n-1}}\left(\log \frac{x_n(1-x_n)^{n-1}2^{n-1}}{\delta}\right)^{n-2} & \text{if } \frac{\delta}{x_n(1-x_n)^{n-1}2^{n-1}} \leq \frac{1}{2} \\ 2^{2-n} n\delta & \text{otherwise} \end{cases} \\ &\leq c_3(n)\delta(\log \frac{1}{\delta})^{n-2}.\end{aligned}$$

Therefore

$$\mathrm{Vol}_n(S \setminus \widetilde{S}(2\delta)) \leq \frac{1}{2^{n-1}(n-1)!}\delta(\log \frac{2^{n-1}}{n^n\delta})^{n-1} + c_1(n)\delta(\log \frac{1}{\delta})^{n-2}$$

$$+ n\int_{1/n}^{1} c_3(n)\delta(\log \frac{1}{\delta})^{n-2}\, dt. \qquad \blacksquare$$

*Remark:* Lemmas 12 and 13 essentially state that the volume of $S \setminus \widetilde{S}(2\delta)$ coincides with that of $W \setminus \widetilde{S}(2\delta)$ up to terms of the order $\delta(\log \frac{1}{\delta})^{n-2}$.

LEMMA 14: *Let $0, e_1, \ldots, e_n$ be the vertices of the simplex $S$, and let $H_1$ and $H_2$ be hyperplanes such that*

(i) $0, e_1, \ldots, e_{n-2} \in H_1, H_2$,

(ii) $e_{n-1} \in \overset{\circ}{H}{}_1^{-}$, $e_n \in \overset{\circ}{H}{}_2^{-}$.

*Then we have for $W = [0, \frac{1}{n}]^n$ and $0 < \delta < \frac{1}{2n!}$*

$$Vol_n\left((W \setminus \widetilde{S}(2\delta)) \cap H_1^+ \cap H_2^+\right) \leq c(n, H_1, H_2)\delta(\log \frac{1}{\delta})^{n-2}.$$

*Proof:* Let the hyperplanes $H_1$ and $H_2$ be given by the equations

$$x_n = a_1 x_{n-1} \qquad \text{and} \qquad x_n = a_2 x_{n-1}.$$

Then

$$\begin{aligned} V := &\ \mathrm{Vol}_n\left((W \setminus \widetilde{S}(2\delta)) \cap H_1^+ \cap H_2^+\right) \\ = &\ \mathrm{Vol}_n\left(0 \leq x_j \leq \frac{1}{n}, \prod_{j=1}^{n} x_j \leq \frac{\delta}{2^{n-1}}, a_1 x_{n-1} \leq x_n \leq a_2 x_{n-1}\right) \\ = &\ n^{-n}\,\mathrm{Vol}_n\left(0 \leq x_j \leq 1, \prod_{j=1}^{n} x_j \leq \frac{\delta n^n}{2^{n-1}}, a_1 x_{n-1} \leq x_n \leq a_2 x_{n-1}\right) \\ = &\ n^{-n}\int_Q \mathrm{Vol}_{n-2}\left(0 \leq x_j \leq 1, j \leq n-2, \prod_{j=1}^{n-2} x_j \leq \frac{\delta n^n}{2^{n-1}st}\right) d(s,t) \end{aligned}$$

where

$$Q = \left\{(s,t) \in [0,1]^2 : a_1 s \leq t \leq a_2 s\right\}.$$

It is easily checked that the set $\{(s,t) \in Q : \frac{\delta n^n}{2^{n-1}st} \geq \frac{1}{2}\}$ has measure at most $\frac{\delta n^n}{2^{n-2}} \log \sqrt{\frac{a_2}{a_1}}$ and for

$$(s,t) \in Q \ s.t. \ \frac{\delta n^n}{2^{n-1}st} \leq \frac{1}{2}$$

we get by Lemma 11

$$\mathrm{Vol}_{n-2}\Big(0 \leq x_j \leq 1, j \leq n-2, \prod^{n-2} x_j \leq \frac{\delta n^n}{2^{n-1}st}\Big) \qquad \leq c_2(n)\frac{\delta}{st}(\log \frac{st}{\delta})^{n-3}$$

Hence

$$V \leq c_1(n,a_1,a_2)\delta + \delta c_2(n) \int_{[st \geq \frac{\delta n^n}{2^{n-2}}] \cap Q} \frac{1}{st}(\log \frac{st}{\delta})^{n-3}\, d(s,t)$$

$$\leq c(n,a_1,a_2)\delta(\log \frac{1}{\delta})^{n-2}. \quad \blacksquare$$

Replacing Schütts' Lemmata 1.3 and 1.4 by our 12, 13 and 14 we get the following modification of Lemma 1.5 ([Schü]).

PROPOSITION 3: *Let $S$ be the simplex spanned by $x_1 = 0, x_2, \ldots, x_{n+1}$. Assume that $S$ has nonempty interior and let $H_1, \ldots, H_n$ be hyperplanes such that*

$$(*) \qquad x_1, \ldots x_{k-1} \in H_k; \quad x_k \in \overset{\circ}{H}{}_k^{+}; \quad x_{k+1}, \ldots, x_{n+1} \in \overset{\circ}{H}{}_k^{-} \quad k = 1, \ldots, n.$$

*Then for sufficiently small $\delta > 0$ we have*

$$\frac{1}{2^{n-1}n!(n-1)!}\delta\Big(\log \frac{2^{n-1}n!}{n^n}\frac{Vol_n(S)}{\delta}\Big)^{n-1} - c_1\delta(\log \frac{1}{\delta})^{n-2}$$

$$\leq \ Vol_n\Big((S \setminus \widetilde{S}(2\delta)) \cap \bigcap_{j=1}^{n} H_j^+\Big)$$

$$\leq \frac{1}{2^{n-1}n!(n-1)!}\delta\Big(\log \frac{2^{n-1}n!}{n^n}\frac{Vol_n(S)}{\delta}\Big)^{n-1} + c_2\delta(\log \frac{1}{\delta})^{n-2}$$

*where $c_1$ and $c_2$ depend on $n$ and on the hyperplanes $H_1, \ldots, H_n$.*

The proofs of the following lemmata can be found in [Schü].

LEMMA 15: *Let $P$ be a convex symmetric polytope in $\mathbb{R}^n$. Then there is a family of simplices $S_i$, $T_i$ $i = 1, \dots, \phi_n(P)$ and hyperplanes $H_x$, $x \in \operatorname{ext}(P)$, such that*

(i) $P \cap H_x^- \cap H_y^- = \phi$ *if* $x \neq y$.

(ii) $\overset{\circ}{S}_i \cap \overset{\circ}{S}_j = \phi$ *if* $i \neq j$ *and* $T_i \supseteq P$ *for all* $i$.

(iii) *For every $i$ there is $x \in \operatorname{ext}(P)$ so that $S_i \subseteq P \cap H_x^-$.*

(iv) *For every $T_i$ there are hyperplanes $H_{ij}, j = 1, \dots, n$ satisfying $(*)$ of Proposition 3 such that*

$$T_i \cap \bigcap_{j=1}^{n} H_{ij}^+ = S_i.$$

(v) *For every $i$ we have that*

$$-T_i \in \{T_k : k = 1, \dots, \phi_n(P)\} \qquad \text{and}$$

$$-S_i \in \{S_k : k = 1, \dots, \phi_n(P)\}.$$

LEMMA 16: *Let $P$ be as above. Then there is a family of simplices $S_i$, $T_i$ $i = 1, \dots, \phi_n(P)$ and hyperplanes $H_{ij}$ $j = 1, \dots, n+1$ so that*

(i) $\overset{\circ}{S}_i \cap \overset{\circ}{S}_j = \phi$ *if* $i \neq j$,

(ii) $\bigcup_{i=1}^{\phi_n(P)} S_i = P$,

(iii) $S_i \subseteq T_i \subseteq P$, $i = 1, \dots, \phi_n(P)$,

(iv) $\bigcap_{j=1}^{n+1} H_{ij}^+ = S_i$, $i = 1, \dots, \phi_n(P)$,

(v) $(H_{ij})_{j=1}^n$ *satisfies $(*)$ of Proposition 3 with respect to $T_i$,*

(vi) $-T_i \in \{T_k : k = 1, \dots, \phi(P)\}$, $-S_i \in \{S_k : k = 1, \dots, \phi_n(P)\}$ *for all* $i = 1, \dots, \phi_n(P)$.

THEOREM 3: *Let $P$ be a convex symmetric polytope in $\mathbb{R}^n$ with nonempty interior. Then we have*

$$\lim_{\delta \to 0} \frac{V_F(0) - V_F(\delta)}{\delta (\log \frac{1}{\delta})^{n-1}} = \frac{1}{n!(n-1)!} \phi_n(P)$$

*where $F$ is the convolution square of $P$ and $V_F$ denotes the distribution function of $F$.*

*Proof:* Clearly the assertion is equivalent to

$$\lim_{\delta \to 0} \frac{\operatorname{Vol}_n \left(P \setminus \frac{1}{2} P(2\delta)\right)}{\delta (\log \frac{1}{\delta})^{n-1}} = \frac{1}{2^{n-1} n!(n-1)!} \phi_n(P).$$

Let $S_i$, $T_i$, $i = 1, \dots, \phi(P)$ be the simplices given by Lemma 15. Since

$$T_i, -T_i \supseteq P \qquad \text{for all} \qquad i = 1, \dots, \phi_n(P)$$

it follows that

$$\begin{aligned}\frac{1}{2}P(2\delta) &= \left\{x \in P : \ \mathrm{Vol}_n\left(P \cap (P + 2x)\right) \geq 2\delta\right\} \\ &\subseteq \left\{x \in T_i : \ \mathrm{Vol}_n\left(T_i \cap (-T_i + 2x)\right) \geq 2\delta\right\} \\ &= \widetilde{T}_i(2\delta) \quad i = 1, \dots, \phi_n(P).\end{aligned}$$

Therefore

$$\begin{aligned}P \setminus \frac{1}{2}P(2\delta) &\supseteq P \setminus \bigcap_{i=1}^{\phi_n(P)} \widetilde{T}_i(2\delta) \\ &\supseteq \bigcup_{j=1}^{\phi_n(P)} S_j \setminus \bigcap_{i=1}^{\phi_n(P)} \widetilde{T}_i(2\delta) \\ &\supseteq \bigcup_{j=1}^{\phi_n(P)} S_j \setminus \widetilde{T}_j(2\delta) \\ &= \bigcup_{j=1}^{\phi_n(P)} (T_j \cap \bigcap_{i=1}^{n} H_{ji}^{+}) \setminus \widetilde{T}_j(2\delta).\end{aligned}$$

Hence

$$\mathrm{Vol}_n\left(P \setminus \frac{1}{2}P(2\delta)\right) \geq \sum_{j=1}^{\phi_n(P)} \mathrm{Vol}_n\left((T_j \setminus \widetilde{T}_j(2\delta)) \cap \bigcap_{i=1}^{n} H_{ji}^{+}\right).$$

We can assume that the only extreme point of $T_j$ which is also an extreme point of $P$ is zero. This follows from the simple observation that

$$\mathrm{Vol}_n\left(\widetilde{T}_j(2\delta)\right) = \mathrm{Vol}_n\left((\widetilde{T_j - x_0})(2\delta)\right).$$

By Proposition 3 (using the right hand side inequality) we have the estimate

$$\begin{aligned}\mathrm{Vol}_n(P \setminus \frac{1}{2}P(2\delta)) \geq & \frac{1}{2^{n-1}n!(n-1)!}\phi_n(P)\delta\left(\log \frac{2^{n-1}n!}{n^n}\frac{\min_j \ \mathrm{Vol}_n(T_j)}{\delta}\right)^{n-1} \\ & - c(n,P)\delta(\log\frac{1}{\delta})^{n-2}.\end{aligned}$$

Since

$$\lim_{\delta\to 0}\frac{\log\frac{c}{\delta}}{\log\frac{1}{\delta}}=1 \qquad \text{for all} \quad c>0$$

we get the desired estimate from below. Using Lemma 16 instead of Lemma 15 we get the same estimate from above. Indeed, let $S_i$, $T_i$, $i=1,\dots,\phi_n(P)$ be the simplices given by Lemma 16, then

$$T_i, -T_i \subseteq P \qquad \forall i=1,\dots,\phi_n(P).$$

Thus

$$\frac{1}{2}P(2\delta)\supseteq \widetilde{T}_i(2\delta) \qquad \forall i=1,\dots,\phi_n(P).$$

We conclude that

$$\begin{aligned}
P\setminus\frac{1}{2}P(2\delta) &\subseteq P\setminus\bigcup_{i=1}^{\phi_n(P)}\widetilde{T}_i(2\delta)\\
&=\bigcup_{j=1}^{\phi_n(P)}S_j\setminus\bigcup_{i=1}^{\phi_n(P)}\widetilde{T}_i(2\delta)\\
&\subseteq\bigcup_{j=1}^{\phi_n(P)}S_j\setminus\widetilde{T}_j(2\delta)\\
&=\bigcup_{j=1}^{\phi_n(P)}(T_j\cap\bigcap_{i=1}^{n+1}H_{ji}^{+})\setminus\widetilde{T}_j(2\delta)\\
&\subseteq\bigcup_{j=1}^{\phi_n(P)}(T_j\cap\bigcap_{i=1}^{n}H_{ji}^{+})\setminus\widetilde{T}_j(2\delta),
\end{aligned}$$

therefore we obtain by Proposition 3

$$\begin{aligned}
\mathrm{Vol}_n\,(P\setminus\frac{1}{2}P(2\delta)) &\leq\sum_{j=1}^{\phi_n(P)}\mathrm{Vol}_n\left((T_j\setminus\widetilde{T}_j(2\delta))\cap\bigcap_{i=1}^{n}H_{ji}^{+}\right)\\
&\leq\frac{1}{2^{n-1}n!(n-1)!}\phi_n(P)\delta\Big(\log\frac{2^{n-1}n!}{n^n}\frac{\max_j\,\mathrm{Vol}_n\,(T_j)}{\delta}\Big)^{n-1}\\
&\quad+c(n,P)\delta(\log\frac{1}{\delta})^{n-2}. \qquad \blacksquare
\end{aligned}$$

*Remark:* It is easy to check that Theorem 3 also holds in the non-symmetric case. ∎

From Theorem 3 and Corollary 2 we immediately get

COROLLARY 6: *Let $P$ be a convex symmetric polytope in $\mathbb{R}^2$, then the distribution function of the convolution square of $P$ determines*

(i) $Vol_2(P)$,

(ii) $Vol_2(P^*)$,

(iii) *the number of extreme points of $P$.*

COROLLARY 7:

(i) *Let $C$ be a convex symmetric body in $\mathbb{R}^2$ such that the distribution function of the convolution square of $C$ is equal to the distribution function of the convolution square of $[-1,1]^2$. Then $C$ is an affine image of $[-1,1]^2$.*

(ii) *If $C$ is a convex body in $\mathbb{R}^n$ such that the distribution function of the convolution square is equal to that of the $n$-dimensional simplex. Then $C$ is an affine image of the simplex.*

Actually (i) and (ii) of Corollary 6 imply that $C$ is an affine image of $[-1,1]^2$, for these are the only convex symmetric bodies that minimize

$$\text{Vol}_2(P) \cdot \text{Vol}_2(P^*).$$

The second assertion of the corollary follows from the fact that for all $n$-dimensional polytopes $P$ $\phi_n(P) \geq (n+1)!$ with equality iff $P$ is a simplex.

*Remarks:* It follows from a theorem of Rogers and Shephard that the simplex in $\mathbb{R}^n$ is also determined by the distribution function of $G = I_C * I_{-C}$. This is because the simplex is the only convex body $C$ in $\mathbb{R}^n$ satisfying

$$\text{Vol}_n\left(C \cap (C+x)\right) = (1 - \|x\|_{S(C)})^n \,\text{Vol}_n C$$

where $S(C) = (C - C)$, i.e. such that equality holds in Lemma 10 (i).

Using the above identity the function $G$ associated with the simplex $S$ can be easily computed:

$$G(x) = \left(1 - \|x\|_{S-S}\right)^n \,\text{Vol}_n(S).$$

Therefore the distribution function of $G$ is given by

$$V_G(\delta) = \left(1 - \left(\frac{\delta}{\text{Vol}_n(S)}\right)^{1/n}\right)^n \binom{2n}{n} \text{Vol}_n(S)$$

and $V_G(0) - V_G(\delta)$ does not behave like $\delta(\log \frac{1}{\delta})^{n-1}$ as $\delta$ tends to zero. However, Theorem 1 provides a tool to determine the polar of the projection body $P_S^*$ of a simplex $S$:

$$\|x\|_{P_S^*} = \lim_{t\to 0} \frac{F(0) - F(tx)}{t}$$
$$= n\|x\|_{S-S} \operatorname{Vol}_n(S).$$

Hence

$$P_S^* = \frac{1}{n \operatorname{Vol}_n(S)}(S - S). \quad ∎$$

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